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Re-examination of the algebraic scattering theory

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Abstract. The foundations of the algebraic scattering theory (AST) are re-examined. This revision leads to a modified version of the AST. It is proven that the algebraic approach alone does not yield a uniquely determined S -matrix. However, a coordinate representation of the generators of the scattering group allows a unique decision about otherwise arbitrary phases. The application of the $SO(2, 3)$ AST and the $SO(1, 3)$ AST to modified Coulomb scattering is reconsidered.

1. Introduction

In the last few years a new approach to the description of scattering phenomena, namely the algebraic scattering theory (AST), has been developed [1-9]. In contrast to the traditional scattering theory, where a potential $V(r)$ has to be specified in order to determine the S -matrix of the corresponding scattering problem, the AST mainly makes use of the symmetry aspects of the problem by employing algebraic techniques in coordinate-free Hilbert spaces. In the fully algebraic version of the theory [7-9], no potential $V(r)$ shows up. Instead, the information about the scattering system is provided by the specification of a scattering group G and the assumption that the Hamiltonian H can be written as a function h of one of the Casimir operators C of the group G : $H = h(C)$. The so-called Euclidean connection [4-9] establishes the link between the interaction zone and the asymptotic, interaction-free region and allows the determination of the S -matrix without reference to explicit coordinates. Henceforward, the S -matrix obtained by means of the AST will be called the algebraic S -matrix. It may depend on the eigenvalues of the operators classifying an irreducible representation of the group G . In the case of the groups $SO(2, 3)$ or $SO(1, 3)$ as the scattering group the algebraic S -matrices were shown to reduce to the S -matrix for Coulomb scattering for special choices of the free parameters [7-9]. These groups were therefore regarded as promising candidates for an algebraic description of modified Coulomb scattering and as the starting point for an algebraic version of coupled-channel calculations [10, 11].

At the early stages of the AST, frequent use was made of realizations of the group operators in certain coordinates. The most important ingredients of the AST were first

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obtained from specific realizations of the operators of the group $SO(2, 1)$ [1-4]. Then it was postulated that all the results still hold for the abstract operators acting on the ket vectors of a Hilbert space, i.e. that all relations be in fact algebraic relations. In the case of one-dimensional scattering this assumption was corroborated by considering different coordinate realizations of the $SO(2, 1)$ group operators [5, 6]. A later investigation of one-dimensional scattering on the basis of the group $SO(2, 2)$ also yielded consistent results [12]. The generalization of the one-dimensional AST to the three-dimensional one was achieved on the level of the fully algebraic theory [7, 8]. No coordinate realization of the operators was written down in the case of the three-dimensional AST. It was still assumed that the results of the three-dimensional AST should hold for any coordinate realization of the theory.

In a recent investigation [13, 14] we examined a specific coordinate realization of the $SO(2, 3)$ AST. Since the cross-sections for typical heavy-ion collisions could be described well by a suitable parametrization of the most general $SO(2, 3)$ S -matrix [9], our main interest was to obtain an expression for the underlying optical potential $V(r)$. Contrary to all expectation the potential we found did not show a Coulomb $1/r$ behaviour for large values of the radial coordinate r . An analytic calculation of the S -matrix belonging to the potential yielded in the framework of traditional scattering theory

$$S_{lv} = \frac{\Gamma[(l+v+\frac{3}{2}-i|\eta|)/2]\Gamma[(l-v+\frac{3}{2}-i|\eta|)/2]}{\Gamma[(l+v+\frac{3}{2}+i|\eta|)/2]\Gamma[(l-v+\frac{3}{2}+i|\eta|)/2]} \frac{\Gamma(i|\eta|)}{\Gamma(-i|\eta|)} \exp\{i[(l+1)\pi-2|\eta|\ln 2]\} \quad (1)$$

whereas the algebraic S -matrix of Alhassid *et al* reads (cf (5.28) of [8])

$$S_{lv}^{AST} = \frac{\Gamma[(l+v+\frac{3}{2}+i\eta)/2]\Gamma[(l-v+\frac{3}{2}+i\eta)/2]}{\Gamma[(l+v+\frac{3}{2}-i\eta)/2]\Gamma[(l-v+\frac{3}{2}-i\eta)/2]} \exp[i(2\eta \ln 2)]. \quad (2)$$

Here, l denotes the quantum number of angular momentum, $\eta = Z_1 Z_2 e^2 \mu / (\hbar^2 k)$ is the Sommerfeld parameter and v a free parameter which allows variation of the potential strength.

Although the structure of the two S -matrices is in principle the same, the differences cannot be accounted for within the AST presented by Alhassid *et al* [1-9]. A preliminary explanation of the origin of the discrepancies has already been outlined in the conclusions of [13]. In this paper we carefully re-investigate the AST and the ambiguities inherent in this theory. We are able to show that the discrepancies disappear in a slightly modified version of the AST. As a consequence of the modifications, we derive in sections 2 and 3 that in the AST two classes of algebraic S -matrices instead of a single one always exist and have to be considered. This means a loss of uniqueness of the S -matrix in the AST. In section 4 we investigate the important case of modified Coulomb scattering and show that—along with the two classes of algebraic S -matrices—the modifications of the AST give rise to a second class of potentials belonging to the same algebraic Hamiltonian.

2. Revision of the one-dimensional AST

In the following we consider only the fully algebraic version of the one-dimensional AST, where the scattering group is taken to be the $SO(2, 1)$ [5, 7]. The commutation

relations for the generators J_+, J_-, J_3 of the $SO(2, 1)$ algebra are given by

$$[J_+, J_-] = -2J_3 \tag{3}$$

$$[J_3, J_\pm] = \pm J_\pm. \tag{4}$$

The scattering states are described by the unitary irreducible representations (unirreps) of $SO(2, 1)$. The basis vectors $|jv\rangle$ of such a representation fulfil:

$$C|jv\rangle = j(j+1)|jv\rangle \tag{5}$$

$$J_3|jv\rangle = v|jv\rangle. \tag{6}$$

Here, $C = -J_+J_- + J_3^2 - J_3$ is the Casimir invariant of the $SO(2, 1)$ algebra. As shown, for example in [15], the spectrum of the allowed values of v in a unirrep can be divided into four classes. One of them, the so-called principal series, describes all the scattering states. In the principal series the eigenvalues of C and J_3 are restricted to the following values:

$$\begin{aligned} j &= -\frac{1}{2} + if & f &\in \mathbb{R} > 0 \\ v &= v_0 + n & n &= 0, \pm 1, \pm 2, \dots & -\frac{1}{2} < v_0 \leq \frac{1}{2}. \end{aligned} \tag{7}$$

In the $SO(2, 1)$ AST one starts with the ansatz $H = h(-(C + \frac{1}{4}))$ for the Hamiltonian. Therefore, H is given by $H = h(f^2)$ in a unirrep of the principal series. Thus, the quantum number $j = -\frac{1}{2} + if$ is connected with the energy. The quantum number v represents a potential strength parameter and allows variation of the depth of the potential. This can easily be seen in a coordinate representation of the operator C [5].

In order to obtain the S -matrix one has to compare the development of the system with interaction to a system which has developed freely. Therefore, in the AST a second group F is introduced which describes the free motion of a particle. In the case of one-dimensional scattering it is natural to choose $F = E(2) \otimes E(1)$ [7, 8]. The $E(2)$ part allows for the freedom connected with the potential strength parameter v , the $E(1)$ part describes the translations of a free particle in one dimension. The $E(2)$ operators are denoted by v_1, v_2, v_3 and fulfil the commutation relations

$$[v_1, v_2] = 0 \tag{8}$$

$$[v_3, v_1] = iv_2 \tag{9}$$

$$[v_3, v_2] = -iv_1. \tag{10}$$

In the AST only such representations are considered for which the $E(2)$ Casimir operator $v_1^2 + v_2^2$ assumes the value 1 [8]. The basis states $|1v\rangle$ of the unirreps of $E(2)$ are chosen to be eigenstates of $v_1^2 + v_2^2$ and v_3 :

$$(v_1^2 + v_2^2)|1v\rangle = 1|1v\rangle \tag{11}$$

$$v_3|1v\rangle = v|1v\rangle. \tag{12}$$

The $E(1)$ operator P acts on the basis states $|\pm k\rangle$ ($k > 0$) of the unirreps of $E(1)$:

$$P|\pm k\rangle = \pm k|\pm k\rangle. \tag{13}$$

The representations of $E(2) \otimes E(1)$ are constructed on the space of the product vectors $|\pm kv\rangle = |1v\rangle \otimes |\pm k\rangle$:

$$P|\pm kv\rangle = \pm k|\pm kv\rangle \tag{14}$$

$$v_3|\pm kv\rangle = v|\pm kv\rangle. \tag{15}$$

In the following we also use the operators $P_{\pm} = P(v_1 \pm iv_2)$ and $L = v_3$ which form a representation of $E(2)$ on the space of the vectors $|\pm kv\rangle$ [8].

For the asymptotic connection of the scattering group G with the free particle group F it is necessary to construct a unirrep of $SO(2, 1)$ on the space of the vectors $|\pm kv\rangle$ (k fixed). Making an ansatz for the $SO(2, 1)$ operators up to second order in the $E(2) \otimes E(1)$ operators P_{\pm}, L ,

$$\begin{aligned} J_3 &= L \\ J_+ &= aP_+ + bP_- + cLP_+ + dLP_- + eP_+^2 + fP_-^2 + gL + hL^2 \\ J_- &= (J_+)^+ \end{aligned} \tag{16}$$

and requiring that the $SO(2, 1)$ commutation relations for J_3 and J_{\pm} hold, one obtains [14]:

$$J_+ = \frac{\exp(i\gamma_{\pm}(k))}{k} [(-\frac{1}{2} + i\delta_{\pm}(k))P_+ + LP_+] \tag{17a}$$

$$J_- = \frac{\exp(-i\gamma_{\pm}(k))}{k} [(+\frac{1}{2} - i\delta_{\pm}(k))P_- + LP_-] \tag{17b}$$

$$J_3 = L. \tag{18}$$

The real parameters $\gamma_{\pm}(k)$ and $\delta_{\pm}(k)$ remain undetermined and may depend on k . The subscript \pm refers to the action of the $SO(2, 1)$ operators on the states $|+km\rangle$ and $|-km\rangle$, respectively.

For the $SO(2, 1)$ Casimir operator $C = -J_+J_- + J_3^2 - J_3$ one obtains

$$C = -\frac{1}{4} - \delta_{\pm}^2(k). \tag{19}$$

Thus the representation (17) and (18) yields the principal series where $j = -\frac{1}{2} + if$. Moreover, $\delta_{\pm}(k)$ is connected with the parameter f via

$$\delta_{\pm}^2 = f^2. \tag{20}$$

It is most important to note that with group theoretical methods it is not possible to specify the connection formulae (17), (18) and (20) any further. The connection formula of Alhassid *et al* (cf (5.10) of [5])

$$J_+ = \frac{\exp(i\gamma_{\pm}(k))}{(\pm k)} [(-\frac{1}{2} \mp if(k))P_+ + LP_+] \tag{21}$$

is not the most general one since here the special choice $\delta_{\pm} = \mp f(k)$ has been made. It will be demonstrated below that only for a specific coordinate representation is it possible to determine the sign with which the parameter f enters into the connection formula.

Before the S -matrix will be derived, we show that the functions $\gamma_{\pm}, \delta_{\pm}$ and f assume definite values in a specific coordinate realization of the operators. We take one of the realizations considered by Alhassid *et al* [5] in the course of the development of a coordinate-free AST. For the group $SO(2, 1)$ we set (cf (3.2) of [5])

$$J_+ = \exp(i\phi) \left[-\frac{\partial}{\partial \rho} + \left(\frac{1}{2} - i \frac{\partial}{\partial \phi} \right) \right] \tag{22}$$

$$J_3 = -i \frac{\partial}{\partial \phi} \tag{23}$$

and for the $E(2) \otimes E(1)$ we take the realization (cf (4.13) of [5])

$$P_{\pm} \equiv Pv_{\pm} = \left(-i \frac{\partial}{\partial \rho} \right) \exp(\pm i\phi) \tag{24}$$

$$L \equiv v_3 = -i \frac{\partial}{\partial \phi}. \tag{25}$$

The basis vectors of the $E(2) \otimes E(1)$ representation space are given by

$$|\pm km\rangle = A_{km}^{\pm} e^{im\phi} e^{\pm ik\rho}. \tag{26}$$

Setting $J_3 = L$ we obtain for J_+ :

$$J_+ = \left[\left(-\frac{1}{2} \frac{\partial}{\partial \rho} \right) + L \right] e^{i\phi}. \tag{27}$$

This expression is in accordance with the most general connection formula (17a)

$$J_+ = \left[\left(-\frac{1}{2} + i\delta_{\pm} \right) + L \right] \frac{e^{i\gamma_{\pm}}}{k} P_+$$

provided one may set

$$\frac{e^{i\gamma_{\pm}}}{k} P_+ = e^{i\phi} \tag{28}$$

when acting on $|\pm km\rangle$. In this case (cf (24)) $\partial/\partial\rho = ik e^{-i\gamma_{\pm}}$ and one obtains

$$\delta_{\pm} = -k e^{-i\gamma_{\pm}}. \tag{29}$$

Combining (24) and (26) one gets

$$P_+ |\pm km\rangle = \pm k e^{i\phi} |\pm km\rangle.$$

Thus, (28) is indeed fulfilled and the phase factor $e^{i\gamma_{\pm}}$ assumes a definite value:

$$e^{-i\gamma_{\pm}} = e^{i\gamma_{\pm}} = \pm 1.$$

From (29) and (20) we get

$$\delta_{\pm} = \mp k = \mp f \quad (f > 0). \tag{30}$$

We conclude that in our specific realization the connection formula for J_+ is given by

$$J_+ = \frac{(\pm 1)}{k} \left[\left(-\frac{1}{2} \mp if \right) P_+ + LP_+ \right].$$

The signs \pm refer to the action of J_+ on the basis states $|+km\rangle$ and $|-km\rangle$, respectively.

In the so-called Euclidean connection one combines the group theoretical connection formulae (17) and (18) with a formula which expresses the vectors $|jv\rangle$ of $SO(2, 1)$ in terms of the vectors $|\pm kv\rangle$ of $E(2)$. The physical idea behind this is that the scattering states $|jm\rangle^{\infty}$ are expressed in terms of the free states $|\pm km\rangle^{\infty}$:

$$|jm\rangle^{\infty} = A_{km} |-km\rangle^{\infty} + B_{km} |+km\rangle^{\infty}. \tag{31}$$

The superscript ∞ indicates that this relation is supposed to hold only in the asymptotic region where the potential has dropped off. Equation (31) is the abstract analogue of the usual development in coordinate space of the scattering state in terms of incoming and outgoing waves [5].

The reflection matrix R_{km} is given by the ratio $R_{km} = B_{km}/A_{km}$. Letting the operators (17) act on the wavefunction (31) and using

$$\begin{aligned}
 J_+|jm\rangle^\infty &= e^{i\beta_m\sqrt{(m-j)(j+m+1)}}|jm+1\rangle^\infty \\
 P_+|+km\rangle^\infty &= e^{i\alpha_m^+k}|+km+1\rangle^\infty \\
 P_+|-km\rangle^\infty &= e^{i\alpha_m^-(-k)}|-km+1\rangle^\infty
 \end{aligned}
 \tag{32}$$

one obtains recursion relations for R_{km} which can be solved analytically:

$$R_{km} = e^{ix_m} e^{im(\gamma_+ - \gamma_-)} \frac{\Gamma(m + \frac{1}{2} + i\delta_+) \Gamma(m_0 + \frac{1}{2} + i\delta_-)}{\Gamma(m + \frac{1}{2} + i\delta_-) \Gamma(m_0 + \frac{1}{2} + i\delta_+)} e^{i\phi(k)}
 \tag{33}$$

where

$$\begin{aligned}
 e^{ix_m} &= (-1)^m \exp\left(i \sum_{l=1}^{m-m_0} (\alpha_{m-l}^+ - \alpha_{m-l}^-)\right) \\
 e^{i\phi(k)} &= (-1)^{m_0} e^{-im_0(\gamma_+ - \gamma_-)} R_{km_0}.
 \end{aligned}$$

Here, R_{km_0} is an entire function of k which can be chosen to be of modulus 1.

Note that in contrast to Alhassid *et al* [5] we do not fix the phases of the step-up and step-down operators in (32). The reason is that we want to make clear that there can appear phase factors in the reflection matrix due to different phase conventions for the ladder operators in (32). For example, the phase convention $P_+|\pm km\rangle = k|\pm km+1\rangle$ leads to an additional factor $(-1)^m$ in R_{km} as compared to the phase convention $P_+|\pm km\rangle = \pm k|\pm km+1\rangle$.

As can be seen from (20), the parameters $\delta_+(k)$ and $\delta_-(k)$ may only differ by a sign:

$$\delta_+ = \delta_- \quad \text{or} \quad \delta_+ = -\delta_-.$$

Only the latter choice yields a non-trivial S -matrix.

The parameter f is positive by convention. So one has to distinguish two distinct classes of reflection matrices R_{km} according to the two different solutions of (20):

$$\delta_+ = +f: R_{km} = e^{ix_m} e^{im(\gamma_+ - \gamma_-)} \frac{\Gamma(m + \frac{1}{2} + if) \Gamma(m_0 + \frac{1}{2} - if)}{\Gamma(m + \frac{1}{2} - if) \Gamma(m_0 + \frac{1}{2} + if)} e^{i\phi(k)}
 \tag{34}$$

$$\delta_+ = -f: R_{km} = e^{ix_m} e^{im(\gamma_+ - \gamma_-)} \frac{\Gamma(m + \frac{1}{2} - if) \Gamma(m_0 + \frac{1}{2} + if)}{\Gamma(m + \frac{1}{2} + if) \Gamma(m_0 + \frac{1}{2} - if)} e^{i\phi(k)}.
 \tag{35}$$

The Euclidean connection reads for the two choices $\delta_+ = +f$ and $\delta_+ = -f$, respectively:

$$\begin{aligned}
 \delta_+ = +f: J_+ &= [(-\frac{1}{2} \pm if) + L] \frac{e^{i\gamma_+}}{k} P_+ \\
 \delta_+ = -f: J_+ &= [(-\frac{1}{2} \mp if) + L] \frac{e^{i\gamma_+}}{k} P_+.
 \end{aligned}$$

For scattering with energy k^2 the relation $H = h(-(C + \frac{1}{4}))$ can be substituted by

$$h(f^2) = k^2$$

when acting on the states $|jm\rangle$ with $j = -\frac{1}{2} + if$. This shows that the function h links the parameter f to the energy k^2 .

Alhassid *et al* get only one class of reflection matrices since they consider only the case $\delta_+ = -f$ (cf equations (5.10) and (6.17) of [5]). This special choice, which turns out to be the correct one for the specific coordinate realization presented above (cf (30)), cannot be justified from a purely algebraic point of view. It cannot be excluded that there exist other coordinate realizations for which the case $\delta_+ = +f$ has to be taken.

In the one-dimensional AST one need not worry about phase conventions for the step-up and step-down operators. Alhassid *et al* noticed the appearance of undetermined k -dependent phase factors in the algebraic S -matrix which do not change the cross section [5, 6, 8]. In fact, the m -dependent phase factors, $\exp(i\chi_m)$ and $\exp(im(\gamma_+ - \gamma_-))$, also drop out in the calculation of the cross section, which is given by the absolute value of the squared reflection amplitude. However, the S -matrices (34) and (35) clearly lead to different cross sections.

We conclude this section by noting that the fully algebraic version of the one-dimensional AST first presented by Alhassid *et al* [7, 8] has to be modified such that always two distinct classes of S -matrices, (34) and (35), are taken into consideration. The sign with which the parameter f enters into the Euclidean connection cannot be determined by means of group theory alone.

3. Discussion of the three-dimensional AST

Up to now three scattering groups have been investigated in the three-dimensional AST: the groups $SO(1, 3)$, $SU(1, 3)$, and $SO(2, 3)$ [6-9]. In this paper we shall mainly be concerned with the $SO(2, 3)$ AST, which is the direct generalization of the one-dimensional $SO(2, 1)$ AST.

The 10 generators L , D , F and V of the $SO(2, 3)$ algebra satisfy the following commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

$$[L_i, V] = 0$$

$$[L_i, D_j] = i\epsilon_{ijk}D_k$$

$$[L_i, F_j] = i\epsilon_{ijk}F_k$$

$$[D_i, F_j] = -i\delta_{ij}V$$

$$[D_i, D_j] = -i\epsilon_{ijk}L_k$$

$$[F_i, F_j] = -i\epsilon_{ijk}L_k$$

$$[D_i, V] = -iF_i$$

$$[F_i, V] = iD_i.$$

Of the two Casimir invariants of the $SO(2, 3)$ algebra only

$$C = L^2 + V^2 - D^2 - F^2$$

is of physical importance. The Casimir operator C is connected with the Hamiltonian H via a function h :

$$H = h(-(C + \frac{3}{4})). \quad (36)$$

The scattering states are represented by the basis states of the unirreps of $SO(2, 3)$ denoted by $|\omega l m v\rangle$:

$$C|\omega l m v\rangle = \omega(\omega + 3)|\omega l m v\rangle$$

$$L^2|\omega l m v\rangle = l(l+1)|\omega l m v\rangle$$

$$L_3|\omega l m v\rangle = m|\omega l m v\rangle$$

$$V|\omega l m v\rangle = v|\omega l m v\rangle$$

with $\omega = -\frac{3}{2} + if$ ($f \in \mathbb{R}^+$). They are classified according to the quantum number ω related by (36) to the energy, according to the quantum number l of angular momentum and m of the z-component of angular momentum, and according to a potential strength parameter v .

As in the case of one-dimensional scattering the unirreps belonging to the principal series ($\omega = -\frac{3}{2} + if$) describe the scattering states completely.

When the Hamiltonian (36) acts on the states $|\omega l m v\rangle$, (36) can be substituted by

$$h(f^2) = k^2$$

where k^2 denotes the scattering energy.

The asymptotic group F is taken to be the product group $E(2) \otimes E(3)$. The $E(3)$ part describes the free motion of a particle in three dimensions and is generated by the momentum and angular momentum operators \mathbf{p} and \mathbf{l} :

$$[p_i, p_j] = 0$$

$$[l_i, l_j] = i\varepsilon_{ijk}l_k$$

$$[l_i, p_j] = i\varepsilon_{ijk}p_k.$$

The two Casimir invariants of $E(3)$ are given by \mathbf{p}^2 and $\mathbf{l}\mathbf{p}$. The $E(2)$ part has already been discussed in the one-dimensional AST. In the three-dimensional AST it is sufficient to consider only those unirreps for which $\mathbf{l}\mathbf{p} = 0$ and $v_1^2 + v_2^2 = 1$ [8]. Thus, the basis states are given by $|\pm k l m v\rangle := |\pm k 0; l m\rangle \otimes |1 v\rangle$ ($k \in \mathbb{R} > 0$), where

$$\mathbf{p}^2|\pm k l m v\rangle = k^2|\pm k l m v\rangle$$

$$l^2|\pm k l m v\rangle = l(l+1)|\pm k l m v\rangle$$

$$l_3|\pm k l m v\rangle = m|\pm k l m v\rangle$$

$$v_3|\pm k l m v\rangle = v|\pm k l m v\rangle.$$

A representation of the $SO(2, 3)$ operators on the space of the vectors $|\pm k l m v\rangle$ is obtained in a similar way as in the case of the one-dimensional AST. We present the results obtained by Wu [6] with a slight but important modification:

$$D = \frac{1}{k} (v_1[\frac{1}{2}(\mathbf{p} \times \mathbf{l} - \mathbf{l} \times \mathbf{p}) + \delta_{\pm} \mathbf{p}] + \frac{1}{2}\{v_2, v_3\} \mathbf{p}) \tag{37}$$

$$F = \frac{1}{k} (v_2[\frac{1}{2}(\mathbf{p} \times \mathbf{l} - \mathbf{l} \times \mathbf{p}) + \delta_{\pm} \mathbf{p}] - \frac{1}{2}\{v_1, v_3\} \mathbf{p})$$

$$L = \mathbf{l}$$

$$V = v_3.$$

Here, $\{ , \}$ denotes the anticommutator, i.e. $\{a, b\} = ab + ba$. In contrast to Wu [6] we write δ_{\pm} instead of f , because—as in the one-dimensional AST—it is not possible to determine in a purely algebraic way with which sign the parameter f enters into the connection (37). It is easily checked that the operators D, F, L and V in (37) satisfy the $SO(2, 3)$ commutation relations. Furthermore, one obtains for the Casimir operator $C = L^2 + V^2 - D^2 - F^2$ acting on the states $|\pm klmv\rangle$:

$$C = -\frac{9}{4} - \delta_{\pm}^2.$$

Thus, the representation (37) of $SO(2, 3)$ belongs to the principal series with

$$\delta_{\pm}^2 = f^2. \tag{38}$$

In the Euclidean connection one combines the representation (37) with an equation which links the basis vectors of the unirreps of $SO(2, 3)$ and $E(2) \otimes E(3)$ in the asymptotic region:

$$|\omega lmv\rangle^{\infty} = A_{lv}|-klmv\rangle^{\infty} + B_{lv}|+klmv\rangle^{\infty}. \tag{39}$$

Letting (37) act on (39) one obtains recursion relations for the reflection coefficients $R_{lv} = B_{lv}/A_{lv}$.

We do not give the technical details here, which can be found in [6]. Alhassid *et al.* work with a Euclidean connection where the substitution $\delta_{\pm} = \pm f$ has already been made. Using a specific phase convention for the step-up and step-down operators they arrive at the result [6, 8]

$$R_{lv} = (-1)^l \frac{\Gamma[(l+v+\frac{3}{2}+if)/2]\Gamma[(l-v+\frac{3}{2}+if)/2]}{\Gamma[(l+v+\frac{3}{2}-if)/2]\Gamma[(l-v+\frac{3}{2}-if)/2]} e^{i\phi(k)}. \tag{40}$$

Here, $e^{i\phi(k)}$ is a k -dependent phase factor which remains undetermined in the algebraic theory. It is worthwhile to note that—as in the one-dimensional AST—different phase conventions for the step-up and step-down operators lead to additional phase factors. We therefore prefer to write the result in a more general form:

$$R_{lv} = e^{i\chi_+ l} \frac{\Gamma[(l+v+\frac{3}{2}+if)/2]\Gamma[(l-v+\frac{3}{2}+if)/2]}{\Gamma[(l+v+\frac{3}{2}-if)/2]\Gamma[(l-v+\frac{3}{2}-if)/2]} e^{i\phi_+(k,v)}. \tag{41}$$

Here, χ_+ is a constant; possible v -dependent phase-factors are absorbed in the function ϕ_+ . The subscript $+$ refers to the choice $\delta_+ = +f$.

The second solution of (38) which yields a non-trivial S -matrix, $\delta_+ = -f$, leads to a different class of S -matrices not mentioned by Alhassid *et al.* Therefore, we modify the algebraic theory so that reflection coefficients of the form

$$R_{lv} = e^{i\chi_- l} \frac{\Gamma[(l+v+\frac{3}{2}-if)/2]\Gamma[(l-v+\frac{3}{2}-if)/2]}{\Gamma[(l+v+\frac{3}{2}+if)/2]\Gamma[(l-v+\frac{3}{2}+if)/2]} e^{i\phi_-(k,v)} \tag{42}$$

are also admitted.

In a typical, simple application of the AST the potential strength is fixed and $\phi(k, v)$ can be regarded as a function of k only. Wu [6] showed that in this case the differential cross section is independent of the k -dependent phase factor $e^{i\phi(k,v)}$. However, the phase factor $e^{i\phi(k,v)}$ shows up in the differential cross section when v is chosen to be complex and l -dependent (see below).

The l -dependent phase factor $e^{i\chi l}$ cannot be determined in a purely algebraic treatment of scattering. It can only be fixed by making some assumptions about the asymptotic behaviour of the wavefunction or of the potential. For example, if one

assumes, like Alhassid *et al.* [6-9], that modified Coulomb potentials can be described in the SO(2, 3) AST, the phases δ_l corresponding to these potentials should behave as $\eta \ln(l+1)$ for large values of l . Here, $\eta = Z_1 Z_2 e^2 \mu / \hbar^2 k$ denotes the Sommerfeld parameter. This expected large- l behaviour has to be compared with that of the phases corresponding to the algebraic S -matrices (41) and (42) (setting $\phi = 0$):

$$\delta_l \sim \frac{\chi_+ l}{2} + f \ln 2 \quad \text{for (41)}$$

$$\delta_l \sim \frac{\chi_- l}{2} - f \ln 2 \quad \text{for (42)}.$$

For positive η one would therefore conclude that the S -matrix (41) with

$$f = \eta \quad \text{and} \quad \chi_+ = 0$$

corresponds to modified Coulomb scattering. This led Alhassid *et al.* to multiply their result (40) for the reflection coefficient, which they got for a specific phase convention for the step-up and step-down operators, by $(-1)^l$ (cf. [6, 8]) in order to obtain the S -matrix $S_{lv} = (-1)^l R_{lv}$.

It is very important to note that such method of arguing may not be consistent with the purely algebraic theory where the asymptotic behaviour of the potential is not known. So, *a priori*, it is not clear whether the ansatz of [6, 8] with $H = \hbar(-C + \frac{9}{4})$ comprises modified Coulomb potentials. In fact, we shall show below that for the group SO(2, 3) there exist realizations of the group theoretical Hamiltonian $H = \hbar(-C + \frac{9}{4})$ for which the corresponding potentials do not show a long-range Coulomb behaviour.

In conclusion, we remark that in a modified version of the three-dimensional AST two distinct classes of S -matrices, (41) and (42), have to be considered. Within one class an algebraic determination of the S -matrix is only possible up to the phase factors e^{ix^l} and $e^{i\phi(k, v)}$. These phase factors may lead to contributions in the differential cross section. They have to be fixed properly and in consistency with the algebraic theory.

4. The three-dimensional AST and modified Coulomb scattering

One of the successes of the AST with SO(2, 3) symmetry was that it could reproduce the S -matrix for Coulomb scattering for a special choice of the free parameters. Choosing $\chi_+ = 0$ and $v = \frac{1}{2}$ in (41) and using $\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1/2-2z}(2\pi)^{1/2}\Gamma(2z)$ one obtains [6, 8, 9]

$$R_{l\ 1/2} = \frac{\Gamma(l+1+if)}{\Gamma(l+1-if)} \exp[i(\phi - 2f \ln 2)]. \tag{43}$$

Thus, setting

$$f = \eta \quad \text{and} \quad \phi = 2f \ln 2 \tag{44}$$

the reflection coefficient (43) just yields the Coulomb phases. With the choice $f = \eta$ the algebraic Hamiltonian H is also determined. From $\hbar(f^2) = \hbar^2 k^2 / 2\mu = E$ one gets $\hbar(x) = E \eta^2 / x$ and

$$H = -\frac{\eta^2}{C + \frac{9}{4}} E. \tag{45}$$

In a recent paper [13] we established the link between the AST and the traditional scattering theory by writing down a specific coordinate realization of the algebraic Hamiltonian. We obtained potentials which did not show a $1/r$ behaviour for large values of the radial coordinate. It resulted in [13]

$$V(r) = \frac{E}{\eta^2} \left[-\frac{(2v)^2 - 1}{4 \cosh^2(kr/|\eta|)} - l(l+1) \left(\frac{1}{(kr/|\eta|)^2} - \frac{1}{\sinh^2(kr/|\eta|)} \right) \right]. \tag{46}$$

An analytical calculation of the S -matrix corresponding to the potentials (46) gave the result

$$S_{l_0} = e^{i(l+1)\pi} \frac{\Gamma[(l+v+\frac{3}{2}-i|\eta|)/2] \Gamma[(l-v+\frac{3}{2}-i|\eta|)/2]}{\Gamma[(l+v+\frac{3}{2}+i|\eta|)/2] \Gamma[(l-v+\frac{3}{2}+i|\eta|)/2]} \times \exp\{2i[\arg \Gamma(i|\eta|) - |\eta| \ln 2]\}. \tag{47}$$

We observe that the S -matrix (47) is in perfect agreement with the modified version of the AST proposed in this paper. Setting $f = |\eta|$, which yields the same form (45) of the algebraic Hamiltonian as the choice $f = \eta$, we see that (47) falls into the second class (42) of algebraic S -matrices not mentioned by Alhassid, Iachello and Wu in their presentation of the $SO(2, 3)$ AST [7-9]. The potential of (46) is one representative of the potentials belonging to this second class of algebraic S -matrices. We remark in passing that in the investigation of one-dimensional scattering problems with an $SO(2, 2)$ group structure [6, 12] Wu *et al* arrived at an S -matrix whose generalization to three dimensions is of the form of (47) (cf. (6.70) of [12]). Among the one-dimensional potentials which can be treated in an $SO(2, 2)$ scattering theory are Pöschl-Teller potentials which are similar to the potential of (46). Furthermore, Wu *et al* noted (e.g. section 7 of [12]) that a generalization of the one-dimensional scattering in a Ginocchio potential to three dimensions would require an $SO(2, 3)$ group structure and would make the potential l -dependent. This indicates that along with our potential (46) there are other potentials in different $SO(2, 3)$ realizations whose S -matrices belong to the second class (42).

Having found a realization of the algebraic Hamiltonian (45) which does not lead to modified Coulomb potentials does not exclude other realizations yielding potentials with a Coulomb tail.

In the case of the three-dimensional AST with the $SO(1, 3)$ as the scattering group the algebraic Hamiltonian corresponding to Coulomb scattering is given by [8, 9]

$$H = -\frac{\eta^2 E}{C+1}. \tag{48}$$

C denotes the $SO(1, 3)$ Casimir invariant $C = L^2 - K^2$, where L and K are the six generators of $SO(1, 3)$. Here, one knows (cf. e.g. [16]) that the Hamiltonian

$$H = -\frac{\eta^2 E}{C+1}$$

can be transformed into the Coulomb Hamiltonian

$$H = \frac{p^2}{2m} + \frac{Z_1 Z_2 e^2}{r} \tag{49}$$

if one takes the realization

$$\begin{aligned} L &= \mathbf{r} \times \mathbf{p} & \mathbf{p} &= -i\nabla \\ K &= \sqrt{\frac{m}{2H}} \left(\frac{1}{2m} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) + \frac{Z_1 Z_2 e^2}{r} \mathbf{r} \right). \end{aligned} \tag{50}$$

In a similar way it may be possible to write down a coordinate realization of the algebraic $SO(2, 3)$ Hamiltonian (45) which leads to the Coulomb Hamiltonian (49).

5. Summary

In this paper the foundations and basic assumptions of the AST were re-examined. It was shown that the formulation of the AST as originally proposed by Alhassid *et al.* [1-9] is not the most general one. It has to be modified in order to be in accordance with the results obtained from a recent specific coordinate realization of the three-dimensional $SO(2, 3)$ AST [13, 14]. In the modified version of the AST derived in this paper, two distinct classes of algebraic S -matrices always have to be considered.

In the modified three-dimensional AST it is only possible to determine the algebraic S -matrix up to an overall k -dependent phase factor and additional phase factors due to the particular phase conventions of the step-up and step-down operators. These phase factors, which may contribute to the cross section, assume definite values for a specific coordinate realization of the algebraic theory. They can only be fixed by providing supplementary information on the scattering system, which goes beyond the framework of a purely algebraic theory.

Finally, we commented on the connection of the algebraic Hamiltonian with the underlying potentials in the case of the three-dimensional AST. We showed that the modifications proposed for the AST, though admitting an additional class of potentials, do not apparently affect the possibility of describing modified Coulomb scattering in the algebraic theory.

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